Matematikk 1

Denne fila inneholder løsningsforslag til følgende ekstra øvingsoppgaver til kapittel 7.10 og 7.11:

7.10.2 a - e
7.10.8
7.10.9
7.10.13 a - c
7.10.14
7.10.19

7.11.1 a, c og g

Delkapittel 7.10:

oppgave 2

a) $f(x,y) = 5x^2y^6$

$$\frac{\partial f}{\partial x} = 5y^6 \cdot 2x = 10xy^6$$

$$\frac{\partial f}{\partial y} = 5x^2\, 6y^5 = 30x^2y^5$$

b) $f(x,y) = \sqrt{x^3 + 3y^4}$

$u = x^3 + 3y^4$

$$\frac{du}{dx} = 3x^2 \qquad \frac{du}{dy} = 12y^3$$

$f(u) = \sqrt{u}$

$$\frac{df}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{df}{dx} = \frac{df}{du}\,\frac{du}{dx} = \frac{1}{2}(x^3+3y^4)^{-1/2} \cdot 3x^2$$

$$= \frac{3x^2}{2\sqrt{x^3+3y^4}}$$

$$\frac{df}{dy} = \frac{df}{du}\,\frac{du}{dy} = \frac{1}{2}(x^3+3y^4)^{-1/2} \cdot 12y^3$$

$$= \frac{6y^3}{\sqrt{x^3+3y^4}}$$

oppgave 2

c) $f(u,v) = \cos(2u+7v)$

$x = 2u+7v$

$\frac{dx}{du} = 2 \qquad \frac{dx}{dv} = 7$

$f(x) = \cos x$

$\frac{df}{dx} = -\sin x$

$\frac{df}{du} = \frac{df}{dx}\frac{dx}{du} = -\sin(2u+7v)\cdot 2$

$= \underline{\underline{-2\sin(2u+7v)}}$

$\frac{df}{dv} = \frac{df}{dx}\frac{dx}{dv} = -\sin(2u+7v)\cdot 7$

$= \underline{\underline{-7\sin(2u+7v)}}$

d) $f(u,v) = \tan(u^2v)$

$x = u^2v$

$\frac{dx}{du} = 2uv \qquad \frac{dx}{dv} = u^2$

$f(x) = \tan(x)$

$\frac{df}{dx} = \frac{1}{\cos^2 x}$

$\frac{df}{du} = \frac{df}{dx}\frac{dx}{du} = \frac{1}{\cos^2(u^2v)}\,2uv = \underline{\underline{\frac{2uv}{\cos^2(u^2v)}}}$

$\frac{df}{dv} = \frac{df}{dx}\frac{dx}{dv} = \frac{1}{\cos^2(u^2v)}\,u^2 = \underline{\underline{\frac{u^2}{\cos^2(u^2v)}}}$

oppgave 2

e) $f(x,y) = e^{3x+y} \ln(xy)$

$f(x.y) = g(x,y)\, h(x.y)$

$g(x.y) = e^{3x+y}$ $\quad$ $h(x,y) = \ln(x.y)$

$u = 3x + y$ $\quad$ $v = x \cdot y$

$\frac{du}{dx} = 3 \quad \frac{du}{dy} = 1$ $\quad$ $\frac{dv}{dx} = y \quad \frac{dv}{dy} = x$

$g(u) = e^{u}$ $\quad$ $h(v) = \ln v$

$\frac{dg}{du} = e^{u}$ $\quad$ $\frac{dh}{dv} = \frac{1}{v}$

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx} = e^{3x+y} \cdot 3$$

$$\frac{dh}{dx} = \frac{dh}{dv}\frac{dv}{dx} = \frac{1}{xy} \cdot y = \underline{\frac{1}{x}}$$

$$\frac{dg}{dy} = \frac{dg}{du}\frac{du}{dy} = e^{3x+y} \cdot 1 = \underline{e^{3x+y}}$$

$$\frac{dh}{dy} = \frac{dh}{dv}\frac{dv}{dy} = \frac{1}{xy} \cdot x = \underline{\frac{1}{y}}$$

$$\frac{df}{dx} = \frac{dg}{dx} \cdot h(x.y) + g(x,y) \cdot \frac{dh}{dx}$$

$$= 3e^{3x+y} \cdot \ln(xy) + e^{3x+y} \cdot \frac{1}{x}$$

$$= \underline{\underline{e^{3x+y}\left(3\ln(xy) + \frac{1}{x}\right)}}$$

$$\frac{df}{dy} = \frac{dg}{dy} h(x,y) + g(x,y) \frac{dh}{dy}$$

$$= e^{3x+y} \ln(xy) + e^{3x+y} \cdot \frac{1}{y}$$

$$= \underline{\underline{e^{3x+y}\left[\ln(xy) + \frac{1}{y}\right]}}$$

Oppgave 8

I et rektangel måles sidekantene til 6 cm (x) og 8 cm (y) og hver sidekant er målt med en feil høyst lik 0,1 cm.

Hva blir arealets maksimale feil?

Arealet av rektangelet er gitt ved:

$z = f(x,y) = x \cdot y$

$f(6,8) = 6 \cdot 8 = \underline{48}\ cm^2$

$f(x+h, y+k) \approx f(x,y) + f_x(x,y) \cdot h + f_y(x,y) \cdot k$

lineær approksimasjon gir

Har at $x = 6$, $y = 8$, $h = k = 0{,}1$. Videre er

$f_x(x,y) = \frac{\partial f}{\partial x} = y \qquad f_x(6,8) = 8$

$f_y(x,y) = \frac{\partial f}{\partial y} = x \qquad f_y(6,8) = 6$

Vi får at:

$$\begin{aligned} f(6+0{,}1,\ 8+0{,}1) &= f(6,8) + f_x(6,8) \cdot 0{,}1 + f_y(6,8) \cdot 0{,}1 \\ &= 48 + 8 \cdot 0{,}1 + 6 \cdot 0{,}1 \\ &= 48 + 0{,}8 + 0{,}6 = 49{,}4\ cm^2 \end{aligned}$$

Arealets maksimale feil blir

$49{,}4 - 48 = \underline{\underline{1{,}4\ cm^2}}$

oppgave 9

Radien og høyden i en rett sylinder måles til 5 cm og 30 cm henholdsvis med en feil høyst lik 0,02 cm for radien og 0,05 cm for høyden.

Hva blir volumets maksimale feil?

$V(r,h) = \pi r^2 \cdot h$

$V(5,30) = \pi \cdot 5^2 \cdot 30 = 750\pi$

Linear approksimasjon gir:

$V(r+k, h+g) = V(r,h) + V_r(r,h) \cdot k + V_h(r,h) g$

der: $r=5$ $h=30$ $k=0{,}02$ og $g = 0{,}05$

$V_r(r,h) = \frac{\partial V}{\partial r} = 2\pi r h$ $\quad V_r(5,30) = 2\pi \cdot 5 \cdot 30 = 300\pi$

$V_h(r,h) = \frac{\partial V}{\partial h} = \pi r^2$ $\quad V_h(5,30) = \pi \cdot 5^2 = 25\pi$

Vi får at:

$$V(5+0{,}02, 30+0{,}05) = 750\pi + 300\pi \cdot 0{,}02 + 25\pi \cdot 0{,}05$$
$$= 750\pi + 6\pi + 1{,}25\pi = 757{,}25\pi \text{ cm}^3$$

Volumets maksimale feil blir

$757{,}25\pi - 750\pi = \underline{\underline{7{,}25\pi \text{ cm}^3}}$

Oppgave 13

Finn $\frac{\partial z}{\partial t}$ ved å bruke kjerneregelen

a) $z = f(x,y) = x^3 - xy \qquad x = t^2$ og $y = 2t+1$

Kjerneregelen sier at:

$$\frac{\partial z}{\partial t} = \underbrace{\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}}_{\frac{\partial m}{\partial t}} + \underbrace{\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}_{\frac{\partial n}{\partial t}}$$

$$\frac{\partial m}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} = (3x^2 - y)\cdot 2t$$
$$= \left(3\left(t^2\right)^2 - (2t+1)\right)\cdot 2t$$
$$= (3t^4 - 2t - 1)\cdot 2t = 6t^5 - 4t^2 - 2t$$

$$\frac{\partial n}{\partial t} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = -x\cdot(2) = -2x$$
$$= -2(t^2) = -2t^2$$

$$\frac{\partial m}{\partial t} + \frac{\partial n}{\partial t} = 6t^5 - 4t^2 - 2t - 2t^2 = \underline{\underline{6t^5 - 6t^2 - 2t}}$$

b) $z = f(x,y) = \cos(4x+5y) \qquad x = t^3 + 3 \qquad y = t^4$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= -\sin(4x+5y)\cdot 4\cdot(3t^2) + (-\sin(4x+5y)\cdot 5)(4t^3)$$
$$= -\sin(4x+5y)(12t^2 + 20t^3)$$
$$= -\sin(4(t^3+3) + 5t^4)(12t^2 + 20t^3)$$
$$= -\sin(5t^4 + 4t^3 + 12)(12t^2 + 20t^3)$$
$$= \underline{\underline{-4t^2(5t+3)\sin(5t^4 + 4t^3 + 12)}}$$

Oppgave 13

c) $z = f(x,y) = \ln(x^3 y^2) \quad x = \sin t \quad y = \cos t$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{1}{x^3y^2} 3x^2y^2 \cos t + \frac{1}{x^3y^2} 2x^3y(-\sin t)$$

$$= \frac{x^2 y}{x^3 y^2}[3y\cos t - 2x\sin t]$$

$$= \frac{1}{xy}[3y\cos t - 2x\sin t]$$

$$= \frac{1}{\sin(t)\cos(t)}[3\cos^2 t - 2\sin^2 t]$$

$$= \frac{3\cos t}{\sin t} - \frac{2\sin t}{\cos t}$$

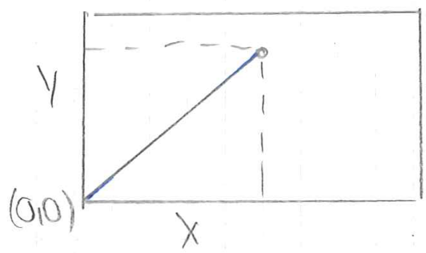
Oppgave 14

En flue kryper på en bordplate og ved tidspunkt t er koordinatene

$$x = t^2 + 4t$$
$$y = t^3 - 3t + 1$$

der meter er enheten for x og y og sekund er enheten for t.

Bestem forandringen per tidsenhet for fluens avstand fra origo når t = 2 s



Fluens avstand fra origo er gitt ved:

$$z = f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\left[2x\cdot(2t+4) + 2y(3t^2-3)\right]$$

$$= (x^2+y^2)^{-\frac{1}{2}}\left(x(2t+4) + y(3t^2-3)\right]$$

Setter t = 2 og får :

$$= (12^2 + 3^2)^{-1/2}(12\cdot 8 + 3\cdot 9)$$

$$= \frac{123}{\sqrt{153}} = \underline{\underline{9{,}94\ m/s}}$$

oppgave 19

Vis at $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

a) $f(x,y) = x^6 y^2 + 4x^2 y + 7$

$\frac{\partial f}{\partial x} = 6x^5 y^2 + 8xy$

$\frac{\partial^2 f}{\partial y \partial x} = \underline{\underline{12x^5 y + 8x}}$

$\frac{\partial f}{\partial y} = 2x^6 y + 4x^2$

$\frac{\partial^2 f}{\partial x \partial y} = \underline{\underline{12x^5 y + 8x}}$

b) $f(x,y) = (5x-3y)^4$

$\frac{\partial f}{\partial x} = 4(5x-3y)^3 \cdot 5$

$= 20(5x-3y)^3$

$\frac{\partial^2 f}{\partial y \partial x} = 60(5x-3y)^2 \cdot (-3)$

$= \underline{\underline{-180(5x-3y)^2}}$

$\frac{\partial f}{\partial y} = 4(5x-3y)^3 \cdot (-3)$

$= -12(5x-3y)^3$

$\frac{\partial^2 f}{\partial x \partial y} = -36(5x-3y)^2 \cdot (5)$

$= \underline{\underline{-180(5x-3y)^2}}$

oppgaver kap 7.11

oppgave 1 a)

$f(x,y) = 2x^2 + y^2 + 4x - 8$

$f_x(x,y) = 4x + 4$

$f_y(x,y) = 2y$

$f_x(x,y) = f_y(x,y) = 0$

$4x + 4 = 0$ $\qquad$ $2y = 0$

$4x = -4$ $\qquad$ $y = 0$

$x = -1$

dvs. når $x = -1$ og $y = 0$

$f_{xx}(x,y) = 4$ $\qquad$ $f_{x,y}(x,y) = 0$

$f_{yy}(x,y) = 2$

$\Delta = f_{xx}(-1,0) \cdot f_{yy}(-1,0) - f_{xy}(-1,0)^2$

$= 4 \cdot 2 - 0 = 8 > 0$

dvs: $\Delta > 0$ og $f_{xx}(-1,0) > 0$, og vi har derfå et lokalt min i punktet $(-1,0)$

$f_{min}(-1,0) = 2 \cdot (-1)^2 + 0^2 + 4(-1) - 8$

$= 2 + 0 - 4 - 8 = -10$

oppgave 1c

$f(x,y) = x^2 - 3y^2 - 6x - 12y - 4$

$f_x(x,y) = 2x - 6$

$f_y(x,y) = -6y - 12$

$f_x(x,y) = f_y(x,y) = 0$

$2x - 6 = 0$ $\qquad$ $-6y - 12 = 0$

$2x = 6$ $\qquad$ $6y = -12$

$x = 3$ $\qquad$ $y = -2$

dvs: når $x=3$ og $y=-2$

$f_{xx}(x,y) = 2$ $\qquad$ $f_{xy}(x,y) = 0$

$f_{yy}(x,y) = -6$

$\Delta = f_{xx}(3,-2) \cdot f_{yy}(3,-2) - f_{xy}(3,-2)^2$

$= 2 \cdot (-6) = -12 < 0$

Siden $\Delta < 0$ gir punktet $(3,-2)$ et sadelpunkt oppgave 1g,

$f(x,y) = x^3 - y^3 + 6x^2 + 6y^2$

$f_x(x,y) = 3x^2 + 12x$
$f_y(x,y) = -3y^2 + 12y$

$f_x(x,y) = f_y(x,y) = 0$

$3x^2 + 12x = 0$
$3x(x+4) = 0$
$3x = 0$
$x = 0$
$x + 4 = 0$
$x = -4$

$-3y^2 + 12y = 0$
$3y(-y+4) = 0$
$3y = 0$
$y = 0$
$-y + 4 = 0$
$y = 4$

dvs: $x = 0 \quad y = 0 \quad (1)$
$x = 0 \quad y = 4 \quad (2)$
$x = -4 \quad y = 0 \quad (3)$
$x = -4 \quad y = 4 \quad (4)$

$f_{xx}(x,y) = 6x + 12$

$f_{xy}(x,y) = 0$

$f_{yy}(x,y) = -6y + 12$

$\Delta_1 = f_{xx}(0,0) \cdot f_{yy}(0,0) - f_{xy}(0,0)^2$
$= 12 \cdot 12 - 0 = 144$

$\Delta_2 = f_{xx}(0,4) \cdot f_{yy}(0,4)$
$= 12 \cdot (-12) = -144$ ⊃: Sadelpunkt

$\Delta_3 = f_{xx}(-4,0) \cdot f_{yy}(-4,0)$
$= -12 \cdot 12 = -144$ ⊃: Sadelpunkt

$\Delta_4 = f_{xx}(-4,4) \cdot f_{yy}(-4,4)$
$= (-12) \cdot (-12) = 144$

$\Delta_1 > 0$ og $f_{xx}(0,0) > 0$ dvs: lokalt min i punktet $(0,0)$

$f_{min}(0,0) = 0$

$\Delta_4 > 0$ og $f_{xx}(-4,4) < 0$ dvs: lokalt max i punktet $(-4,4)$

$f_{max}(-4,4) = 64$