Delkapittel 13.1

oppgave 2

b) Vi skal finne taylorpolynomet av grad 1, 2 og 3 for

$$f(x) = x^3 - 3x^2 + 5x - 2 \quad \text{om} \quad a = 0$$

Taylorpolynomet P_n av grad n om punktet $x = a$ for f er gitt ved:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$+ \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Vi har at:

$f(x) = x^3 - 3x^2 + 5x - 2$ $\qquad$ $f(0) = -2$

$f'(x) = 3x^2 - 6x + 5$ $\qquad$ $f'(0) = 5$

$f''(x) = 6x - 6$ $\qquad$ $f''(0) = -6$

$f'''(x) = 6$ $\qquad$ $f'''(0) = 6$

Vi får at:

$$P_1(x) = f(a) + f'(a)(x-a) = f(0) + f'(0)(x-0)$$

$$= \underline{-2 + 5x}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$= \underline{-2 + 5x - 3x^2}$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3$$

$$= \underline{-2 + 5x - 3x^2 + x^3}$$

Oppgave 2 e)

$$f(x) = \sin^{-1} x \qquad \text{om} \qquad x = 0$$

Har at:

$$f(x) = \sin^{-1} x = \arcsin x \qquad f(0) = 0$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \qquad f'(0) = 1$$

$$f''(x) = -\frac{1}{2}(1-x^2)^{-\frac{1}{2}-1} \cdot (-2x)$$

$$= x(1-x^2)^{-\frac{3}{2}} \qquad f''(0) = 0$$

$$f'''(x) = 1 \cdot (1-x^2)^{-\frac{3}{2}} + x\left(-\frac{3}{2}\right)(1-x^2)^{-\frac{5}{2}} \cdot (-2x)$$

$$= (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}$$

$$f'''(0) = 1$$

Får at:

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= 0 + 1 \cdot x = \underline{x}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= 0 + 1 \cdot x + \frac{0}{2} \cdot x^2 = \underline{x}$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= 0 + 1 \cdot x + \frac{0}{2} \cdot x^2 + \frac{1}{6} x^3$$

$$= \underline{x + \frac{1}{6} x^3}$$

oppgave 3c)

Vi skal finne taylorpolynomet av grad 3 for

$f(x) = \frac{1}{x}$ om $a = 1$

Har at:

$f(x) = \frac{1}{x}$ $\quad f(1) = \frac{1}{1} = 1$

$f'(x) = \frac{0 \cdot x - 1 \cdot 1}{x^2} = \frac{-1}{x^2}$ $\quad f'(1) = -1$

$f''(x) = \frac{0 \cdot x^2 - (-1) \cdot 2x}{x^4} = \frac{2}{x^3}$ $\quad f''(1) = 2$

$f'''(x) = \frac{0 \cdot x^3 - 2 \cdot 3x^2}{x^6} = \frac{-6}{x^4}$ $\quad f'''(1) = -6$

Får at:

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= 1 - 1(x-1) + \frac{2}{2}(x-1)^2 + \frac{-6}{6}(x-1)^3$$

$$= \underline{1 - (x-1) + (x-1)^2 - (x-1)^3}$$

Kap. 13.2

oppgave 1 : konvergens, divergens, grenseverdi

a) $a_n = 2 + \left(\frac{3}{5}\right)^n$

Betrakter følgen som en funksjon $f(x)$

$$f(x) = 2 + \left(\frac{3}{5}\right)^x$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 2 + \underbrace{\left(\frac{3}{5}\right)^x}_{\text{går mot null når } x \to \infty} = 2$$

dvs: $\lim_{n \to \infty} 2 + \left(\frac{3}{5}\right)^n = 2$ $\quad$ ɔ: konvergerer

d) $a_n = \frac{\sin n}{n^2 + 3}$

$$f(x) = \frac{\sin x}{x^2 + 3}$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sin x}{x^2 + 3} = 0$$

Teller $|\sin x| \leq 1$ mens nevner $\to \infty$

dvs: $\lim_{n \to \infty} \frac{\sin n}{n^2 + 3} = 0$ $\quad$ ɔ: Konvergerer

j) $a_n = \frac{3n^3 - 2}{4n^2 + 7n}$

$$f(x) = \frac{3x^3 - 2}{4x^2 + 7x} \qquad \lim_{x \to \infty} f(x) = \frac{\infty}{\infty} = \lim_{x \to \infty} \frac{9x^2}{8x + 7}$$

$$= \frac{\infty}{\infty} = \lim_{x \to \infty} \frac{18x}{8} = \infty$$

ɔ: $\{a_n\}$ divergerer

oppgave 5

b) $a_n = \frac{n+2}{n+1}$

$a_1 = \frac{1+2}{1+1} = \frac{3}{2}$

$a_2 = \frac{2+2}{2+1} = \frac{4}{3}$

$a_3 = \frac{3+2}{3+1} = \frac{5}{4}$

$S_3 = \frac{3}{2} + \frac{4}{3} + \frac{5}{4}$

$= \frac{18}{12} + \frac{16}{12} + \frac{15}{12} = \frac{49}{12}$

oppgave 6

b) $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \ldots\ldots$

$= \sum_{n=0}^{\infty} \frac{1}{(4n+1)} \cdot (-1)^n$

c) $\frac{1}{6} + \frac{4}{7} + \frac{9}{8} + \frac{16}{9} + \frac{25}{10} + \ldots$

$= \sum_{n=1}^{\infty} \frac{n^2}{n+5}$

Delkapittel 13.2

Oppgave 7 a)

Vi skal finne summen til den geometriske rekken

$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots$$

En geometrisk rekke er definert ved

$$a + ak + ak^2 + ak^3 + \dots = \sum_{n=0}^{\infty} ak^n$$

En geometrisk rekke konvergerer dersom $|k| < 1$ med en sum gitt ved:

$$S = \sum_{n=0}^{\infty} ak^n = a + ak + ak^2 + ak^3 + \dots = \frac{a}{1-k}$$

I vårt tilfelle er $a = 1$ og $k = \frac{1}{5}$

Siden $k = \frac{1}{5} < 1$, så konvergerer rekken med følgende sum:

$$S = \frac{a}{1-k} = \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \underline{\underline{\frac{5}{4}}}$$

d) $\frac{1}{4} - \frac{1}{10} + \frac{1}{25} - \frac{2}{125} + \dots$

$a = \frac{1}{4}$

For å finne k kan vi dele et ledd på det foregående.

Eks: første ledd er a og andre ledd er $a \cdot k$

$$\frac{ak}{a} = k \quad \text{dvs:} \quad \frac{-\frac{1}{10}}{\frac{1}{4}} = -\frac{1}{10} \cdot \frac{4}{1}$$

$$= -\frac{4}{10} = -\frac{2}{5}$$

dvs $k = -\frac{2}{5}$

$$S = \frac{a}{1-k} = \frac{\frac{1}{4}}{1 + \frac{2}{5}}$$

$$= \frac{1}{4} \cdot \frac{5}{7} = \underline{\underline{\frac{5}{28}}}$$

Delkapittel 13.3

oppgave 1 a)

$$e^{-2x} = 1 - \frac{2}{1!}x + \frac{2^2}{2!}x^2 - \dots = \sum (-1)^n \frac{2^n}{n!} x^n$$

Vi har fra teorem 13.7 at:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Vi erstatter x med -2x og får:

$$e^{-2x} = 1 + \frac{(-2x)}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \dots$$

$$= 1 - \frac{2}{1!}x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \dots = \sum (-1)^n \frac{2^n}{n!} x^n$$

Kap 13.3

Oppgave 1d)

Teorem 13.7

$$(1+2x)e^x = (1+2x)\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$= (1+2x) + \frac{(x+2x^2)}{1!} + \frac{(x^2+2x^3)}{2!} + \frac{(x^3+2x^4)}{3!} + \cdots$$

$$= 1 + \frac{2x}{1!} + \frac{x}{1!} + \frac{4x^2}{2!} + \frac{x^2}{2!} + \frac{6x^3}{3!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(2n+1)x^n}{n!}$$

Oppgave 2a

Def. 13.5

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

kalles taylorrekken til funksjonen f om punktet $x=a$

$a=0$

$f(x) = 3^x$ $\qquad f(0) = 1$

$f'(x) = 3^x \ln 3$ $\qquad f'(0) = \ln 3$

$f''(x) = 3^x (\ln 3)^2$ $\qquad f''(0) = (\ln 3)^2$

$f'''(x) = 3^x (\ln 3)^3$ $\qquad f'''(0) = (\ln 3)^3$

$\vdots$

$$3^x = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$= 1 + \ln 3 \cdot (x-0) + \frac{(\ln 3)^2}{2!}(x-0)^2 + \frac{(\ln 3)^3}{3!}(x-0)^3 + \dots$$

$$= 1 + (\ln 3)x + \frac{(\ln 3)^2}{2!}x^2 + \frac{(\ln 3)^3}{3!}x^3 + \dots$$

oppgave 2 b

$a = 0$

$f(x) = \sin^2 x = (\sin x)^2$ $\quad f(0) = 0$

$f'(x) = 2 \sin x \cdot \cos x$ $\quad f'(0) = 0$

$f''(x) = 2(\cos x \cdot \cos x + \sin x(-\sin x))$ $\quad f''(0) = 2$

$= 2(\cos^2 x - \sin^2 x)$

$f'''(x) = 4 \cos x \cdot (-\sin x) - 4 \sin x \cdot \cos x$

$= -8 \cos x \sin x$ $\quad f'''(0) = 0$

$f''''(x) = -8((-\sin x)\sin x) + \cos x \cdot \cos x)$

$= -8(\cos^2 x - \sin^2 x)$ $\quad f''''(0) = -8 = -2^3$

$$\sin^2 x = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$= 0 + 0(x-0) + \frac{2}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 - \frac{2^3}{4!}(x-0)^4 + \dots$$

$$= x^2 - \frac{2^3}{4!}x^4 + \dots$$

Oppgave 6c

Regler

$\ln(a \cdot b) = \ln a + \ln b$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$$f(x) = \ln(5 + x)$$

$$= \ln\left(5 \cdot \left(1 + \frac{1}{5}x\right)\right)$$

$$= \ln 5 + \ln\left(1 + \frac{1}{5}x\right)$$

erstatter x med $\frac{1}{5}x$ i teorem 13.8

$$f(x) = \ln 5 + \left[\frac{1}{5}x - \frac{\left(\frac{1}{5}x\right)^2}{2} + \frac{\left(\frac{1}{5}x\right)^3}{3} - \frac{\left(\frac{1}{5}x\right)^4}{4} + \dots\right]$$

$$= \ln 5 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{5}x\right)^n}{n} \rightarrow \left(\frac{1}{5}\right)^n \cdot x^n = \frac{1^n}{5^n} x^n = \frac{x^n}{5^n}$$

$$= \ln 5 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{5^n \cdot n}$$

Oppgave 7

Vi skal finne taylorrekken for hver f om a og deretter skrive rekken v.h.a summetegn.

a) $f(x) = \ln x \qquad a = 1$

Definisjon 13.5 sier:

La en funksjon f være uendelig mange ganger deriverbar i et åpent intervall som inneholder $x=a$

Rekken

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

kalles taylorrekken til funksjonen f om punktet $x=a$

ln er uendelig deriverbar og vi får at:

$f(x) = \ln x \qquad f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{0 \cdot x - 1 \cdot 1}{x^2} = \frac{-1}{x^2} \qquad f''(1) = \frac{-1}{1} = -1$

$f'''(x) = \frac{0 \cdot x^2 - (-1) \cdot 2x}{x^4} = \frac{2x}{x^4} = \frac{2}{x^3} \qquad f'''(1) = \frac{2}{1} = 2$

Taylorrekken for $f(x) = \ln x$ om $a=1$ er gitt ved:

$$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$= 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4 + \dots$$

$$= \frac{1}{1}(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

$$+ (-1)^{N-1}\frac{(x-1)^N}{N}$$

$$= \underline{\underline{\sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-1)^n}{n}}}$$

Kan også bruke teorem 13.8 direkte ved å erstatte X med (X-1)

Oppgave 10d

Vi skal finne maclaurinrekken for f ved å bruke teorem 13.9 og deretter skrive rekken ved hjelp av summetegn.

$$f(x) = \sqrt[4]{1+x} = (1+x)^{\frac{1}{4}}$$

En annen maclaurinrekke er binomialrekken

Teorem 13.9 sier at:

Binomialrekken konvergerer mot den funksjonen den representerer for $|x| < 1$

$$(1+x)^{\alpha} = 1 + \begin{bmatrix}\alpha \\ 1\end{bmatrix} x + \begin{bmatrix}\alpha \\ 2\end{bmatrix} x^2 + \dots = \sum_{n=0}^{\infty} \begin{bmatrix}\alpha \\ n\end{bmatrix} x^n$$

der α er en reell konstant og

$$\begin{bmatrix}\alpha \\ 0\end{bmatrix} = 1 \quad \begin{bmatrix}\alpha \\ n\end{bmatrix} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \quad \text{for } n \geq 1$$

Vi har at:

$\alpha - n + 1$ gir:

$$\binom{1/4}{0} = 1$$

$$\binom{1/4}{1} = \frac{1/4}{1!} = \frac{1}{4}$$

$$\left(\frac{1}{4} - 1 + 1 = \frac{1}{4}\right)$$

$$\binom{1/4}{2} = \frac{1/4 \cdot (-3/4)}{2!}$$

$$\left(\frac{1}{4} - 2 + 1 = -\frac{3}{4}\right)$$

$$= \frac{1 \cdot (-3)}{4 \cdot 4 \cdot 2!}$$

$$\binom{1/4}{3} = \frac{1/4 \cdot (-3/4) \cdot (-7/4)}{3!}$$

$$\left(\frac{1}{4} - 3 + 1 = -\frac{7}{4}\right)$$

$$= \frac{1 \cdot (-3) \cdot (-7)}{4 \cdot 4 \cdot 4 \cdot 3!}$$

$$\binom{1/4}{4} = \frac{1/4 \cdot (-3/4) \cdot (-7/4) \cdot (-\frac{11}{4})}{4!}$$

$$\left(\frac{1}{4} - 4 + 1 = -\frac{11}{4}\right)$$

$$= \frac{1 \cdot (-3) \cdot (-7) \cdot (-11)}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4!}$$

Prøver å sette opp binomialrekken:

$$(1+x)^{\frac{1}{4}} = 1 + \frac{1}{4}x - \frac{1 \cdot 3}{4^2(2!)}x^2 + \frac{1 \cdot 3 \cdot 7}{4^3(3!)}x^3 - \frac{1 \cdot 3 \cdot 7 \cdot 11}{4^4 \cdot (4!)}x^4 + \dots$$

$(4n-5)$

$n=2$ gir $(8-5)=3$

$n=3$ gir $(12-5)=7$

$n=4$ gir $(16-5)=11$

$n=5$ gir $(20-5)=15$

o.s.v

$$= 1 + \frac{1}{4}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)\, x^n}{4^n\,(n!)}$$